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Integral representation of fermionic Schwinger functions

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Abstract. The families of fermionic Schwinger functions for the free theory and for theories interacting with a classical gauge field are constructed as (a subset of) the moments of certain measures, in close analoguey to the well known bosonic case. The fermionic measures, however, are not uniquely determined and can be chosen, for example, as non-centred Gaussian measures with support on a Hilbert–Schmidt extension of the two-particle Hilbert space $\mathcal{H} \wedge \mathcal{H}$.

1. Introduction

A Euclidean quantum field theory is given by the set of its Schwinger functions, which allow a reconstruction of the corresponding Wightman quantum field theory in Minkowski spacetime [24]. For interacting boson models probabilistic methods for constructing Schwinger functions are of major importance [6], but cannot be applied to fermionic theories due to the asymmetry of their Schwinger functions. Although Segal's guage space approach [25, 26] gives non-commutative analogues of probabilistic notions and has been widely used for fermions [7], no success comparable to the bosonic case has been obtained with this method.

The most common approach in dealing with fermionic asymmetry is the algebraic integration theory of Berezin [3], but which invalidates all classical probabilistic techniques of bosonic theory. Kree [14, 15] has developed a unified picture of bosonic and fermionic integration as an algebraic calculus of forms.

In this paper a new probabilistic method for representing fermionic Schwinger functions as moments of a measure is developed. The key observation is the fact that, due to its $SL(2, \mathbb{C})$ transformation properties, only Schwinger functions with an even number of fermionic arguments are different from zero [28, theorem II.6]. Therefore, one may restrict attention to the even part of Euclidean fermionic Fock space.

The second section of this paper introduces the mathematical concepts needed in the following, beginning with asymmetric tensor algebra over the one-particle Hilbert space $\mathcal{H}$. The exponential of a second-order asymmetric tensor is examined and, finally, triplets of Hilbert spaces $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ are mentioned, which are of great importance for the construction of the measure spaces used in this paper.

The third section discusses different integral representations of Schwinger functions of free Euclidean Fermi fields. Of these possibilities the case of a non-centred Gaussian measure is used in the last section to implement the interaction with a classical electromagnetic field.

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This paper also shows the applicability of classical probabilistic methods to fermionic theories. In future work, which is currently in progress, the problem of implementing more non-trivial fermionic models as perturbations of the measures obtained in this paper will be addressed. This is the main motivation for this paper, which will serve as a starting point for further work. The measure theoretic point of view was chosen since bosonic theories have already shown the power (but also high degree of difficulty) of this ansatz.

2. Algebraic aspects of fermionic integration

Before going into details it should be stressed that the term *algebraic* is used in a rather restricted sense, referring only to asymmetric tensor algebra over one particle space. This paper does not address any questions related to operator algebras for representing CAR. An excellent reference for this subject is a paper by Araki [1].

2.1. Basic Hilbert space and asymmetric forms

Let $\mathcal{E}$ be a complex separable Hilbert space and $\mathcal{E}^* \cong \mathcal{E}$ its dual space. Let $\mathcal{H} = \mathcal{E} \oplus \mathcal{E}^*$ be the direct sum of $\mathcal{E}$ and $\mathcal{E}^*$. $\mathcal{H}$ is again a complex Hilbert space with inner product $(f|g)$, which is assumed to be linear in the second argument. $\mathcal{H}$ has the following additional structure.

(1) There exists an antiunitary involution

$$\mathcal{H} \ni f \mapsto f^* \in \mathcal{H} \qquad f^{**} = f \tag{1}$$

such that $\mathcal{E}$ is isometrically mapped to $\mathcal{E}^*$ and vice versa.

(2) The duality between $\mathcal{E}$ and $\mathcal{E}^*$ can be extended to the bilinear form

$$\langle f|g\rangle = (f^*|g) \qquad f, g \in \mathcal{H} \tag{2}$$

on $\mathcal{H}$, such that $\mathcal{E}$ and $\mathcal{E}^*$ are maximal isotropic subspaces of this form.

The basic example is $\mathcal{H} = \mathcal{L}^2(\mathbb{R}) \otimes \mathbb{C}^2$ with the familiar inner product $(f|g) = \int_{\mathbb{R}} f^\dagger(x) g(x)\,\mathrm{d}x$. $\mathcal{E}$ and $\mathcal{E}^*$ are given by

$$\mathcal{E} = \mathcal{L}^2(\mathbb{R}) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathcal{E}^* = \mathcal{L}^2(\mathbb{R}) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with the involution * defined by

$$f^*(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{f}(x).$$

Besides the symmetric form (2), an asymmetric form

$$\omega : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$$

can be constructed as follows. Let $j_\pm : \mathcal{H} \mapsto \mathcal{H}$ be given by

$$j_\pm(a + b) = a \pm b \qquad a \in \mathcal{E}, b \in \mathcal{E}^*. \tag{3}$$

Then $\langle f|j_+g\rangle$ is the symmetric form (2) and

$$\omega(f, g) = \langle f|j_-g\rangle \qquad f, g \in \mathcal{H} \tag{4}$$

is an asymmetric form on $\mathcal{H}$. Similarly, there can be defined asymmetric forms for more general operators than j_-. For that purpose let $M : \mathcal{E} \mapsto \mathcal{E}$ be a bounded linear operator on

$\mathcal{E}$. The asymmetric extension M_- of M to a bounded linear operator on $\mathcal{H}$ can be defined by

$$\begin{aligned} M_- f &= Mf \qquad f \in \mathcal{E} \\ M_- f &= -(M^\dagger f^*)^* \qquad f \in \mathcal{E}^*. \end{aligned} \tag{5}$$

The bilinear form $\omega_M : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$

$$\omega_M(f, g) = \langle f | M_- g \rangle \qquad f, g \in \mathcal{H} \tag{6}$$

is clearly asymmetric in f and g and is uniquely determined by its values for $f, g \in \mathcal{E}$ by

$$\omega_M(f^*, g) = (f | Mg)$$

since $\mathcal{E}$ and $\mathcal{E}^*$ are isotropic subspaces.

For an arbitrary asymmetric bilinear form ω on $\mathcal{H}$ one can always find a maximal isotropic decomposition $\mathcal{H} = \mathcal{F} \oplus \mathcal{G}$ with $\mathcal{F} \cong \mathcal{G}$. Then, at least after a unitary transformation, $\mathcal{F}$ may be identified with $\mathcal{E}$ and also $\mathcal{G}$ with $\mathcal{E}^*$. Therefore, equation (6) gives the most general kind of asymmetric bilinear forms on $\mathcal{H}$.

2.2. Antisymmetric tensor algebra

Let $\mathcal{H}^{\otimes n}$ be the n-fold tensor product

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$$

with inner product

$$(f_1 \otimes \cdots \otimes f_n | g_1 \otimes \cdots \otimes g_n) = n! \prod_{k=1}^{n} (f_k | g_k). \tag{7}$$

Observe the factor $n!$ in front of the right-hand side of equation (7).

The asymmetric tensor product of elements $f_1, \ldots, f_n \in \mathcal{H}$ is defined by

$$f_1 \wedge \cdots \wedge f_n = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}. \tag{8}$$

The incomplete space of algebraic tensors of rank n over $\mathcal{H}$ is denoted by $\mathcal{A}_n^-(\mathcal{H})$ and the completion with respect to the inner product (7) by $\mathcal{T}_n^-(\mathcal{H})$.

The inner product (7) in $\mathcal{T}_n^-(\mathcal{H})$ is deliberately chosen to be normalized to the following determinant

$$\| f_1 \wedge \cdots \wedge f_n \|_n^2 = \det((f_i | f_j)). \tag{9}$$

The Hilbert space direct sum of the $\mathcal{T}_n^-(\mathcal{H})$—the fermionic Fock space—is denoted by $\mathcal{T}^-(\mathcal{H})$, that is

$$\mathcal{T}^-(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{T}_n^-(\mathcal{H}) \tag{10}$$

where $\mathcal{T}_0^-(\mathcal{H})$ simply means $\mathbb{C}$. The inner product in $\mathcal{T}^-(\mathcal{H})$ is defined by

$$\| F \|^2 = \sum_{n=0}^{\infty} \| F_n \|_n^2 \tag{11}$$

if $F = \sum_{n=0}^{\infty} F_n$, $F_n \in \mathcal{T}_n^-(\mathcal{H})$. A dense subset of $\mathcal{T}^-(\mathcal{H})$ is given by the algebraic sum

$$\mathcal{A}^-(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{A}_n^-(\mathcal{H})$$

of all elements $F=\sum_{n=0}^{\infty}F_n$, $F_n\in\mathcal{A}_n^-(\mathcal{H})$, with $F_n=0$ for all sufficiently large n.

The involution (1) and the bilinear form (2) on $\mathcal{H}$ can be naturally extended to $\mathcal{T}^-(\mathcal{H})$ by

$$(f_1\wedge\cdots\wedge f_n)^*=f_n^*\wedge\cdots\wedge f_1^*,\qquad f_i\in\mathcal{H}, \mathrm{i}=1,\ldots,n$$

and similarly

$$\langle A|B\rangle=(A^*|B)\qquad A,B\in\mathcal{T}^-(\mathcal{H}).$$

Let $C_2(\mathcal{H})$ be the set of all Herbert–Schmidt (HS) operators on $\mathcal{H}$. Then it is well known [17] that there is a one-to-one correspondence between tensors $A\in\mathcal{H}\wedge\mathcal{H}$ and HS operators $L_A\in C_2(\mathcal{H})$ on $\mathcal{H}$, given by

$$\begin{aligned}&\mathcal{H}\wedge\mathcal{H}\ni A\mapsto L_A\in C_2(\mathcal{H})\\&\langle A|f\wedge g\rangle=\langle f|L_Ag\rangle.\end{aligned}\tag{12}$$

If $A\in\mathcal{E}^*\wedge\mathcal{E}$, then one even has $L_Af\in\mathcal{E}\forall f\in\mathcal{E}$ and also $L_Ag\in\mathcal{E}^*\forall g\in\mathcal{E}^*$, that is, L_A can be considered as an HS operator on the space $\mathcal{E}$.

The even part $\mathcal{T}_{\text{even}}^-(\mathcal{H})$ of the fermionic Fock space is defined by

$$\mathcal{T}_{\text{even}}^-(\mathcal{H})=\sum_{n=0}^{\infty}\mathcal{T}_{2n}^-(\mathcal{H}).$$

Coherent states (in the sense given, for example, in the introduction of the book [13]) in $\mathcal{T}_{\text{even}}^-(\mathcal{H})$ are defined by

$$\exp Z=1+Z+\frac{1}{2}Z\wedge Z+\frac{1}{3!}Z\wedge Z\wedge Z+\cdots\qquad Z\in\mathcal{D}\tag{13}$$

where $\mathcal{D}$ is some dense subspace of $\mathcal{H}\wedge\mathcal{H}$, may be equal to $\mathcal{H}\wedge\mathcal{H}$. Since

$$\exp f\wedge g=1+f\wedge g\Leftrightarrow f\wedge g=\exp f\wedge g-\exp 0$$

for $f,g\in\mathcal{H}$, it is easy to see that the linear hull of the coherent states (13) is dense in $\mathcal{T}_{\text{even}}^-(\mathcal{H})$.

The convergence of $\exp Z$ in $\mathcal{T}_{\text{even}}^-(\mathcal{H})$ is shown in the following lemma.

Lemma 1. Let $A_1,A_2\in\mathcal{H}\wedge\mathcal{H}$. Then

$$\langle\exp A_1|\exp A_2\rangle=\det{}_{\mathcal{H}}(\mathbb{1}+L_{A_1}L_{A_2})^{\frac{1}{2}}.\tag{14}$$

If one even has $A_1,A_2\in\mathcal{E}^*\wedge\mathcal{E}$, and so L_{A_1},L_{A_2} can be considered as operators on $\mathcal{E}$, one obtains

$$\langle\exp A_1|\exp A_2\rangle=\det{}_{\mathcal{E}}(\mathbb{1}+L_{A_1}L_{A_2}).\tag{15}$$

Proof. Equation (15) is well known, for example, from [17]. After some obvious notational changes the general form (14) can be identified with Theorem III.7 in the work of Jaffe, Lesniewski and Weitsman [12]. □

The symbols $\det_{\mathcal{H}}(\mathbb{1}+L_{A_1}L_{A_2})$ and $\det_{\mathcal{E}}(\mathbb{1}+L_{A_1}L_{A_2})$ denote the determinant of $\mathbb{1}+L_{A_1}L_{A_2}$ as operator on the space $\mathcal{H}$ and, respectively, $\mathcal{E}$. Note that in the bosonic case the exponential $\exp B=1+B+\frac{1}{2}B\vee B+\cdots$ does not converge for all $B\in\mathcal{H}\vee\mathcal{H}$ (see, for example, [18, equation 3.25]).

For inner products of tensors $A \in \mathcal{T}^-(\mathcal{H})$ with exponentials of second order tensors one obtains the following formula, which clearly shows the asymmetric Gaussian combinatorics of free fermionic $2n$-point functions.

$$\begin{aligned}\langle \exp A | f_1 \wedge \cdots \wedge f_{2n} \rangle &= \frac{1}{n!} \langle A^{\wedge n} | f_1 \wedge \cdots \wedge f_{2n} \rangle \\ &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma) \langle A | f_{\sigma(1)} \wedge f_{\sigma(2)} \rangle \cdots \langle A | f_{\sigma(2n-1)} \wedge f_{\sigma(2n)} \rangle \qquad (16)\end{aligned}$$

with $f_i \in \mathcal{H}, i = 1, \ldots, 2n$. If one has $A \in \mathcal{E}^* \wedge \mathcal{E}$, then (16) considerably simplifies to [17]

$$\begin{aligned}\langle \exp A | f_n^* \wedge \cdots \wedge f_1^* \wedge g_1 \wedge \cdots \wedge g_n \rangle &= \sum_{\sigma \in Sn} \operatorname{sign}(\sigma) \langle A | f_1^* \wedge g_{\sigma(1)} \rangle \cdots \langle A | f_n^* \wedge g_{\sigma(n)} \rangle \\ &= \sum_{\sigma \in Sn} \epsilon(\sigma) (f_1 | L_A g_{\sigma(1)}) \cdots (f_n | L_A g_{\sigma(n)}) \\ &= (f_1 \wedge \cdots \wedge f_n | \Gamma(L_A)(g_1 \wedge \cdots \wedge g_n))\end{aligned}$$

where $f_i, g_j \in \mathcal{E}, i, j = 1, \ldots, n$. Here $\Gamma(L) : \mathcal{T}^-(\mathcal{H}) \mapsto \mathcal{T}^-(\mathcal{H})$ denotes the usual second quantization of a bounded linear operator $L : \mathcal{H} \mapsto \mathcal{H}$, defined by

$$\Gamma(A)(f_1 \wedge \cdots \wedge f_n) = (Af_1) \wedge \cdots \wedge (Af_n) \qquad f_i \in \mathcal{H}.$$

For an operator $L_A : \mathcal{E} \mapsto \mathcal{E}$, which, according to equation (12), is defined by an element $A \in \mathcal{E}^* \wedge \mathcal{E}$, one consequently obtains

$$L_{\exp A} = \Gamma(L_A). \qquad (17)$$

The left-hand side of this equation is defined in analoguey to (12) by

$$\langle \exp A | F^* \wedge G \rangle = (F | L_{\exp A} G) \qquad F, G \in \mathcal{T}^-(\mathcal{H}).$$

2.3. Hilbert space triplets

Let $T : \mathcal{H} \mapsto \mathcal{H}$ be an HS operator, such that $\mathcal{H}_1 = T\mathcal{H}$ is dense in $\mathcal{H}$. $\mathcal{H}_1$ itself is a Hilbert space with inner product

$$(f|g)_1 = (T^{-1} f | T^{-1} g) \qquad f, g \in \mathcal{H}_1.$$

Let $\mathcal{H}_{-1}$ be the completion of $\mathcal{H}$ with respect to the inner product

$$(f|g)_{-1} = (Tf | Tg) \qquad f, g \in \mathcal{H}.$$

Obviously, one has $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ and $\mathcal{H}_{-1}$ is the dual space of $\mathcal{H}_1$ with respect to the duality which is defined by the inner product in $\mathcal{H}$ [2]. Note that [2, section 1.9]

$$(\mathcal{H} \wedge \mathcal{H})_{\pm 1} = \mathcal{H}_{\pm 1} \wedge \mathcal{H}_{\pm 1}.$$

3. Construction of measures

3.1. $\mathcal{T}^-_{\mathrm{even}}(\mathcal{H})$ as $\mathcal{L}^2$-space

The coherent states (13) in $\mathcal{T}^-_{\mathrm{even}}(\mathcal{H})$ allow for the construction of the mapping

$$A \ni \mathcal{T}^-_{\mathrm{even}}(\mathcal{H}) \mapsto \phi_A(Z) = (\exp Z | A) \qquad Z \in \mathcal{D} \qquad (18)$$

from $\mathcal{T}^-_{\text{even}}(\mathcal{H})$ to the space $\mathcal{F}(\mathcal{D})$ of functions $\phi_A(Z)$ over $\mathcal{D}$, where $\mathcal{D}$ is the subspace of $\mathcal{H}\wedge\mathcal{H}$ mentioned after equation (13). This space becomes a reproducing kernel Hilbert space (see [22], but also [20]) with respect to the scalar product

$$(\phi_F\|\phi_G) = (F|G) \qquad F, G \in \mathcal{T}^-_{\text{even}}(\mathcal{H}). \tag{19}$$

The reproducing kernel is given by

$$K(A, B) = (\exp A|\exp B)$$

that is

$$(K(\cdot, Z)\|\phi_F) = \phi_F(Z).$$

The space $\mathcal{F}(\mathcal{D})$ can be represented as a subspace of an $\mathcal{L}^2$-space. For that purpose, let $(\mathcal{H}\wedge\mathcal{H})_1 \subset \mathcal{H}\wedge\mathcal{H} \subset (\mathcal{H}\wedge\mathcal{H})_{-1}$ be the Hilbert space triplet constructed from a HS operator T in $\mathcal{H}$ as in section 2.3. Let $(\mathcal{H}\wedge\mathcal{H})^{\mathbb{R}}$ and $(\mathcal{H}\wedge\mathcal{H})^{\mathbb{R}}_{\pm 1}$ be the real subspaces with respect to the conjugation (1), that is, the spaces of all $A = A^*$. Furthermore, let $\mu_{0,\mathbb{1}}$ be complex white noise on $(\mathcal{H}\wedge\mathcal{H})_{-1}$ and $\nu_{0,\mathbb{1}}$ be real white noise on $\mathrm{i}(\mathcal{H}\wedge\mathcal{H})^{\mathbb{R}}_{-1}$, the imaginary subspace of $(\mathcal{H}\wedge\mathcal{H})_{-1}$, that is, the subspace of all $A^* = -A$, which again is a real Hilbert space. For a general discussion of the support properties of measures on infinite dimensional spaces see [29]. Complex white noise is constructed in the book of Hida [9].

Let $F \in \mathcal{T}^-_{\text{even}}(\mathcal{H}_1)$ be such that in the representation

$$F = \sum_{n=0}^{\infty} F_{2n} \qquad F_{2n} \in \mathcal{T}^-_{2n}(\mathcal{H}_1)$$

only a finite number of elements F_{2n} is different from zero. Then $\phi_F(Z)$ is a continuous function on $(\mathcal{H}\wedge\mathcal{H})_{-1}$ and $\mathrm{i}(\mathcal{H}\wedge\mathcal{H})^{\mathbb{R}}_{-1}$. The choice $\mathrm{i}(\mathcal{H}\wedge\mathcal{H})^{\mathbb{R}}_{-1}$ for the support of real white noise is motivated by the fact that for a real orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, that is, a basis which fulfils

$$e_i^* = e_i \qquad \forall i \in \mathbb{N} \tag{20}$$

the basic monomials $(\exp Z|e_i\wedge e_j) = (Z|e_i\wedge e_j)$ are real functions. But this choice is only a matter of convenience and each other real subspace of $\mathcal{H}_{-1}\wedge\mathcal{H}_{-1}$ would also suffice.

Due to the linearity properties of integral and scalar product it is sufficient to consider elements of the form

$$F = e_{i_1}\wedge\cdots\wedge e_{i_{2n}} \qquad i_1 < \cdots < i_{2n}$$
$$G = e_{j_1}\wedge\cdots\wedge e_{j_{2n}} \qquad j_1 < \cdots < j_{2n}$$

where all e_i are elements of the real basis (20). Then one obtains

$$\begin{aligned}(\exp Z|F) &= \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \operatorname{sign}(\sigma)(Z|e_{i_{\sigma(1)}}\wedge e_{i_{\sigma(2)}})\cdots(Z|e_{i_{\sigma(2n-1)}}\wedge e_{i_{\sigma(2n)}})\\ &= \sum_{\sigma\in OS_{2n}} \operatorname{sign}(\sigma)(Z|e_{i_{\sigma(1)}}\wedge e_{i_{\sigma(2)}})\cdots(Z|e_{i_{\sigma(2n-1)}}\wedge e_{i_{\sigma(2n)}})\end{aligned}$$

and similarly for $(\exp Z|G)$. Let the set of ordered permutations OS_{2n} be defined by

$$OS_{2n} = \{\sigma\in S_{2n}|\sigma(2i-1) < \sigma(2i), \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)\}$$

with $\sharp\{OS_{2n}\} = \frac{(2n)!}{2^n n!} = (2n-1)!!$. For $\rho\in\{\nu_{0,\mathbb{1}}, \mu_{0,\mathbb{1}}\}$, then, one gets

$$\begin{aligned}&\int \overline{(\exp Z|F)}(\exp Z|G)\,\mathrm{d}\rho(Z)\\ &\quad= \sum_{\sigma,\pi\in OS_{2n}} \operatorname{sign}(\sigma)\operatorname{sign}(\pi)\int \overline{(Z|e_{i_{\sigma(1)}}\wedge e_{i_{\sigma(2)}})}\cdots(Z|e_{j_{\pi(2n-1)}}\wedge e_{j_{\pi(2n)}})\,\mathrm{d}\rho(Z).\end{aligned}$$

Since $(Z|e_i \wedge e_j)$ and $(Z|e_k \wedge e_l)$ are independent Gaussian random variables for $\{i, j\} \neq \{k, l\}$, only the case $\sigma = \pi$ for $F = G$ gives a non-vanishing contribution and one obtains

$$\int \overline{(\exp Z|F)}(\exp Z|G)\,\mathrm{d}\rho(Z)$$

$$= \sum_{\sigma\in OS_{2n}} \int |(Z|e_{i_{\sigma(1)}} \wedge e_{i_{\sigma(2)}})|^2\,\mathrm{d}\rho(Z)\cdots\int |(Z|e_{i_{\sigma(2n-1)}} \wedge e_{i_{\sigma(2n)}})|^2\,\mathrm{d}\rho(Z)$$

$$= \sum_{\sigma\in OS_{2n}} 1 = (2n-1)!!. \tag{21}$$

Let $M : \mathcal{T}^-_{\mathrm{even}}(\mathcal{H}) \mapsto \mathcal{T}^-_{\mathrm{even}}(\mathcal{H})$ be defined by $M|_{\mathcal{T}^-_{2n}(\mathcal{H})} = \frac{1}{\sqrt{(2n-1)!!}}$, that is

$$MF = \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n-1)!!}} F_{2n} \qquad F = \sum_{n=0}^{\infty} F_{2n},\ F_{2n} \in \mathcal{T}^-_{2n}(\mathcal{H})$$

M is self-adjoint and $M1 = 1$. The mapping

$$\mathcal{T}^-_{\mathrm{even}}(\mathcal{H}_1) \ni F \mapsto \Psi_F(Z) = \phi_{MF}(Z) = (\exp Z|MF) \tag{22}$$

is an isometry from $\mathcal{T}^-_{\mathrm{even}}(\mathcal{H}_1)$ to a subspace of $\mathcal{L}^2((\mathcal{H}\wedge\mathcal{H})_{-1}, \mu_{0,\mathbb{1}})$ and also $\mathcal{L}^2(\mathrm{i}(\mathcal{H}\wedge\mathcal{H})^{\mathbb{R}}_{-1}, \nu_{0,\mathbb{1}})$, respectively.

$$\int \overline{\Psi_F(Z)}\Psi_G(Z)\mathrm{d}\rho(Z) = (F|G) \qquad \forall F, G \in \mathcal{T}^-_{\mathrm{even}}(\mathcal{H}_1). \tag{23}$$

The image of $\mathcal{T}^-_{\mathrm{even}}(\mathcal{H}_1)$ under the mapping (22) is a true subspace of both $\mathcal{L}^2$-spaces, since each coordinate $(Z|e_i \wedge e_j)$ in $\Psi_F(Z)$ appears at most linearly. Because of (23), Ψ_F can be extended to all $F \in \mathcal{T}^-_{\mathrm{even}}(\mathcal{H})$ and gives an isomorphism

$$\mathcal{T}^-_{\mathrm{even}}(\mathcal{H}) \ni F \mapsto \Psi_F(Z) = (\exp Z|MF) \in \mathcal{L}^2(\mathrm{i}(\mathcal{H}_{-1}\wedge\mathcal{H}_{-1})^{\mathbb{R}}, \nu_{0,\mathbb{1}}) \tag{24}$$

between $\mathcal{T}^-_{\mathrm{even}}(\mathcal{H})$ and a true subspace of $\mathcal{L}^2(\mathrm{i}(\mathcal{H}_{-1}\wedge\mathcal{H}_{-1})^{\mathbb{R}}, \nu_{0,\mathbb{1}})$.

But this isomorphism only respects the linear structure of $\mathcal{T}^-_{\mathrm{even}}(\mathcal{H})$. The algebraic structure as even part of the Grassmann algebra over $\mathcal{H}$ cannot be mapped onto numeric multiplication of the functions $\phi_F(Z)$ or $\Psi_F(Z)$. One has to introduce the new product

$$\Psi_F(Z) * \Psi_G(Z) = \Psi_{F\wedge G}(Z). \tag{25}$$

This product, however, cannot be reduced in any case to a numeric identity between $\mathcal{L}^2$ functions.

3.2. Schwinger functions of the free Dirac field

The Schwinger functions of a free fermionic quantum field theory are obtained by

$$\langle \exp \Omega_S | f_1 \wedge \cdots \wedge f_{2n}\rangle = S_{2n}(f_1, \cdots, f_{2n}) \qquad f_i \in \mathcal{H}_1 \tag{26}$$

where $\Omega_S \in \mathcal{H}_{-1}\wedge\mathcal{H}_{-1}$ is the tensor which represents the two point function

$$S_2(f, g) = \langle \Omega_S | f \wedge g\rangle \qquad f, g \in \mathcal{H}_1. \tag{27}$$

A sufficient condition for the existence of Ω_S is continuity of S_2 on $\mathcal{H}$ ([17], lemma 3).

Without loss of generality one may assume that S_2 is given in the form

$$S_2(f, g) = \langle f | S_- g\rangle$$

where, according to (6), S_- is constructed from a continuous operator S on $\mathcal{E}$. So the tensor Ω_S and the operator S on $\mathcal{E}$ are uniquely determined by each other. If S is positive it can be absorbed into the definition of the scalar product in $\mathcal{E}$. In that case Ω_S is given by the canonical tensor

$$\Omega_- = \sum_{\mu=1}^{\infty} e_\mu^* \wedge e_\mu.$$

If S is not positive, then, according to a slightly modified version of theorem 2, section 1.2, in [4], there exists a uniquely determined positive sesquilinear form β and an anti-unitary operator J on $\mathcal{H}$ with $J^2 = -\mathbb{1}$, such that

$$S_2(f, g) = \beta(Jf|g).$$

The operator J, in general, explicitly depends on the operator S on $\mathcal{E}$. This procedure is used in [17–19] for a moment representation of Schwinger functions, but with a different function space representation of the complete fermionic Fock space.

Another possibility to obtain a positive operator is the following construction of formal doubling of degrees of freedom, which, in the present form, can be found in Kupsch [19], where the connection to the well known construction of eight-component Euclidean Dirac spinors of Osterwalder and Schrader [23] is also given. From two copies $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ of $\mathcal{H}$ one forms the space

$$\hat{\mathcal{H}} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$$

and analogueously for $\hat{\mathcal{E}} = \mathcal{E}^{(1)} \oplus \mathcal{E}^{(2)}$ and $\hat{\mathcal{E}}^* = \mathcal{E}^{(1)*} \oplus \mathcal{E}^{(2)*}$. Then, $\hat{\mathcal{H}}$ is decomposed into

$$\hat{\mathcal{H}} = \hat{\mathcal{E}} \oplus \hat{\mathcal{E}}^*.$$

If the polar decomposition of the operator $S : \mathcal{E} \mapsto \mathcal{E}$ is given by

$$S = UA_1 = A_2 U$$

with a unitary operator U and positive operators A_1, A_2, then let $\hat{S}$ on $\hat{\mathcal{E}}$ be defined by

$$\hat{S} = \begin{pmatrix} A_2 & S \\ S^\dagger & A_1 \end{pmatrix}. \tag{28}$$

$\hat{S}$ is easily seen to be positive, but has the non-trivial kernel

$$K := \mathrm{Ker} S = \{\hat{f} = (Uf, -f) | f \in \mathcal{E}\}$$

which is isomorphic to $\mathcal{E}$. On the factor space $\hat{F} := \hat{\mathcal{H}}/(K \oplus K^*)$ the form $\omega_{\hat{S}}$ according to (6) is equivalent to S_2. The positive form β and the operator J from above are

$$\beta(\hat{f}, \hat{g}) = (\hat{f}|\hat{S}_+\hat{g})$$
$$J\hat{f} = \begin{cases} -\hat{f}^* & \text{for } \hat{f} \in \hat{\mathcal{E}} \\ \hat{f}^* & \text{for } \hat{f} \in \hat{\mathcal{E}}^*. \end{cases}$$

On $\hat{F}$, the operator $\hat{S}$ can be absorbed into the scalar product, and in this case Ω_S is again given by the canonical tensor Ω_-.

For other than free fields the Schwinger functional still has the form

$$\langle F_S | f_1 \wedge \cdots \wedge f_{2n} \rangle \qquad F_S \in \mathcal{T}^-_{\mathrm{even}}(\mathcal{H}_{-1}),\ f_i \in \mathcal{H}_1 \tag{29}$$

but the tensor F_S no longer has the simple form $F_S = \exp \Omega_S$. A general discussion of the functional (29) in a supersymmetric context can be found in the work of Haba and Kupsch [8].

For the Dirac field one has

$$\mathcal{E} = \mathcal{L}^2(\mathbb{R}^4) \otimes \mathbb{C}^4.$$

For an element $f \in \mathcal{H} = \mathcal{E} \oplus \mathcal{E}^* \cong \mathcal{L}^2(\mathbb{R}^4) \otimes \mathbb{C}^8$, let $\mathcal{E}$ correspond to the upper four components and $\mathcal{E}^*$ to the lower ones.

The Euclidean Dirac operator is $(m - \not{\partial})$, with inverse operator

$$S = (m - \not{\partial})^{-1}. \tag{30}$$

It is supposed that $m > 0$. The HS operator T on $\mathcal{E}$ for the construction of the triplet $\mathcal{E}_1 \subset \mathcal{E} \subset \mathcal{E}_{-1}$ can be chosen as

$$T = (1 + x^2 - \triangle)^{-2}$$

with the Laplace operator $\triangle = \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}$. Then S is a continuous operator on $\mathcal{E}, \mathcal{E}_{\pm 1}$. The polar decomposition $S = AU$ with a positive operator A and a unitary operator U is given by

$$\begin{aligned} A &= \sqrt{S^\dagger S} = (m^2 - \triangle)^{-\frac{1}{2}} \\ U &= (m^2 - \triangle)^{-\frac{1}{2}}(m - \not{\partial}). \end{aligned} \tag{31}$$

The involution in $\mathcal{H}$ is

$$\mathcal{H} \ni f \mapsto f^*(x) = \begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix} \overline{f}(x)$$

where $\mathbb{1}_4$ is the four-by-four unit matrix. The two-point Schwinger function $S_2(f, g)$ is given by

$$\begin{aligned} \omega(f, g) &= \int f^T(x) \begin{pmatrix} 0 & S(x, y) \\ -S^T(y, x) & 0 \end{pmatrix} g(y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \langle \Omega_S | f \wedge g \rangle = \langle f | S_- g \rangle. \end{aligned}$$

S is not positive [23, section 3.2]. This can be overcome with the previously described doubling of degrees of freedom.

For a massless Dirac particle the operator S on $\mathcal{E}$ is unbounded, but if one chooses

$$T = (-\triangle)^{\frac{1}{2}}(1 + x^2 - \triangle)^{-2}$$

as the HS operator, then S is not only bounded but even nuclear on $\mathcal{E}_1$ [19, section 7].

The operators S (30) and A (31) fulfil the following commutation relations, where $\gamma_5 := -\gamma_1\gamma_2\gamma_3\gamma_4$.

$$\begin{aligned} S\gamma_5 &= -\gamma_5 S \\ A\gamma_5 &= \gamma_5 A. \end{aligned}$$

The projection operators onto right- and left-handed Weyl spinors are

$$P_\pm = \tfrac{1}{2}(\mathbb{1} \pm \gamma_5)$$

and give a decomposition of $\mathcal{E}$ into

$$\mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$$

where $\mathcal{E}_\pm = P_\pm \mathcal{E}$. The operators $P_\pm$ can be extended to the whole space $\mathcal{H}$ by

$$P_\pm f = \begin{pmatrix} P_\pm & 0 \\ 0 & P_\mp \end{pmatrix} \qquad \forall f \in \mathcal{H}.$$

Then S_2 can be decomposed into

$$S_2(f, g) = S_2^+(f, g) + S_2^-(f, g) \qquad \forall f, g \in \mathcal{H} \tag{32}$$

with $S_2^{\pm}(f, g) = S_2(f, P_{\pm} g)$ ([18], section 7). An analogueous decomposition can be obtained for the theory with doubled degrees of freedom. For that purpose let the projection operators $\hat{P}_{\pm}$ on $\hat{\mathcal{E}}$ be defined by

$$\hat{P}_{\pm} = \tfrac{1}{2} \begin{pmatrix} P_{\mp} & 0 \\ 0 & P_{\pm} \end{pmatrix}.$$

Again $\hat{P}_{\pm}$ can be extended to the whole space $\hat{\mathcal{H}}$ by $\hat{P}_{\pm} f = (\hat{P}_{\pm} f^*)^*$ for $f \in \hat{\mathcal{E}}^*$. This gives a decomposition of $\omega_{\hat{S}}(f, g) := \langle f | \hat{S}_- g \rangle$ into

$$\omega_{\hat{S}}(f, g) = \omega_{\hat{S}}^{+}(f, g) + \omega_{\hat{S}}^{-}(f, g) \tag{33}$$

with $\omega_{\hat{S}}^{\pm}(f, g) = \omega_{\hat{S}}(f, \hat{P}_{\pm} g)$ [19, section 7].

Further discussions of the two point functions of Dirac, Weyl and Majorana spinors can be found in [21].

3.3. Integral representation of Schwinger functionals

One is looking for measures ρ on $\mathcal{H}_{-1} \wedge \mathcal{H}_{-1}$, or a subspace of it, such that one has a representation

$$\langle \exp \Omega_S | A \rangle = \int \langle \exp Z | A \rangle \mathrm{d}\rho(Z). \tag{34}$$

The measure ρ is not uniquely determined by (34), since, due to the asymmetric tensor multiplication, each coordinate $\langle Z | e_i \wedge e_j \rangle, i < j$, appears at most, linearly and so only the zeroth and first moments of each monomial are determined.

3.3.1. Delta measure. The simplest solution is given by the Dirac delta measure δ_{Ω_S} supported on $\Omega_S \in T_2^{-}(\mathcal{H}_{-1})$. In this case, equation (34) becomes a trivial identity. S is not required to fulfill any positivity conditions. Since Dirac measures in different points are always mutually singular, this possibility is not further examined.

3.3.2. Induced measure. The second solution is given by a measure which is induced by complex white noise $\mu_{0,\mathbb{1}}^{\mathcal{E}}$ on $\mathcal{E}_{-1}$. For this version one needs a positive operator S, since it works only for the canonical tensor Ω_-.

One considers the isotropic decomposition of $\mathcal{H}_1$ into $\mathcal{E}_1 \oplus \mathcal{E}_1^*$ with respect to the asymmetric bilinear form Ω_-. One obtains an orthonormal basis $\{e_\mu\}_{\mu \in \mathbb{N}}$ in $\mathcal{E}_1$ such that $\langle \Omega_- | e_\mu^* \wedge e_\nu \rangle = \delta_{\mu\nu}$. Let the real subspace $\mathcal{D} \subset (\mathcal{H}_1 \wedge \mathcal{H}_1)^{\mathbb{R}}$ be given as the linear span of all vectors $e_\mu^* \wedge e_\mu$, $\mu \in \mathbb{N}$, that is

$$\mathcal{D} = \mathcal{L}\{e_\mu^* \wedge e_\mu | \mu \in \mathbb{N}\}$$

and define P_D to be an orthogonal projection onto $\mathcal{D}$.

For each element $A \in \mathcal{H}_1 \wedge \mathcal{H}_1$ there exists a $B \in \mathcal{H} \wedge \mathcal{H}$ with $A = \Gamma_2(T) B$. Therefore

$$L_A = T L_B T^{\dagger} \tag{35}$$

such that for all $A \in \mathcal{H}_1 \wedge \mathcal{H}_1$ the operator L_A is trace class.

On the real subspace $(\mathcal{H}_1 \wedge \mathcal{H}_1)^{\mathbb{R}}$ let the functional χ be defined by

$$\begin{aligned} \chi(A) &= \sum_{\mathcal{E}_{-1}} \mathrm{e}^{\mathrm{i}\langle P_D A | \phi^* \wedge \phi \rangle} \, \mathrm{d}\mu_{0,\mathbb{1}}^{\mathcal{E}} = \sum_{\mathcal{E}_{-1}} \mathrm{e}^{\mathrm{i}(\phi | L_{P_D A} \phi)} \, \mathrm{d}\mu_{0,\mathbb{1}}^{\mathcal{E}}(\phi) \\ &= \det_{\mathcal{E}}(\mathbb{1} - \mathrm{i} L_{P_D A})^{-1} \qquad \forall A \in (\mathcal{H}_1 \wedge \mathcal{H}_1)^{\mathbb{R}}. \end{aligned} \tag{36}$$

For the existence of χ it is essential that $L_{P_D A}$ is trace class [27].

It is not difficult to show that $A_n \to A$ in $\mathcal{H}_1 \wedge \mathcal{H}_1$ implies $\mathrm{Tr}\,|L_{A_n}| \to \mathrm{Tr}\,|L_A|$, and from continuity of determinant relative to trace norm the continuity of the functional (36) in the norm of $\mathcal{H}_1 \wedge \mathcal{H}_1$ follows. From its construction χ is positive definite and so is a characteristic functional on $\mathcal{H}_1 \wedge \mathcal{H}_1$.

From a HS operator $T : \mathcal{H}_1 \mapsto \mathcal{H}_1$ one can construct the triplet $\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_{-2}$ and the functional (36), then, defines a probability measure ρ on $(\mathcal{H}_{-2} \wedge \mathcal{H}_{-2})^{\mathbb{R}}$ [29]. Since $\chi(A) = 1$ for $A \in D^{\perp}$, the measure ρ is supported on $(\mathcal{D})' \subset (\mathcal{H}_{-2} \wedge \mathcal{H}_{-2})^{\mathbb{R}}$ [29, part A, theorem 19.1].

The functional (36) has the power series

$$\det_{\mathcal{E}}(\mathbb{1} - \mathrm{i}L_{P_D A})^{-1} = 1 + \mathrm{i}\,\mathrm{Tr}_{\mathcal{E}}\, L_{P_D A} + \cdots. \tag{37}$$

For an orthonormal basis $\{e_\mu\}_{\mu\in\mathbb{N}}$ in $\mathcal{E}$ one has

$$\begin{aligned}\mathrm{Tr}_{\mathcal{E}}\, L_{P_D A} &= \sum_{\mu\in\mathbb{N}} (e_\mu | L_{P_D A} e_\mu) = \sum_{\mu\in\mathbb{N}} ((P_D A)^{\dagger} | e_\mu^* \wedge e_\mu) \\ &= \sum_{\mu\in\mathbb{N}} (e_\mu^* \wedge e_\mu | A) = (\Omega_- | A).\end{aligned}$$

So the first moment of ρ is given by

$$\int \langle Z|A\rangle\, \mathrm{d}\rho(Z) = (\Omega_-|A) = \langle \Omega_-|A\rangle. \tag{38}$$

In Dirac notation $L_{e_\mu^* \wedge e_\mu} = |e_\mu\rangle\langle e_\mu|$, and so it is easily seen that

$$L_{e_\mu^* \wedge e_\mu} L_{e_\nu^* \wedge e_\nu} = \delta_{\mu\nu} L_{e_\mu^* \wedge e_\mu}.$$

For $\mu \neq \nu$ one consequently obtains

$$(\mathbb{1} - \mathrm{i}L_{e_\mu^* \wedge e_\mu})(\mathbb{1} - \mathrm{i}L_{e_\nu^* \wedge e_\nu}) = (\mathbb{1} - \mathrm{i}(L_{e_\mu^* \wedge e_\mu} + L_{e_\nu^* \wedge e_\nu})) \tag{39}$$

and so

$$\begin{aligned}\chi(L_{e_\mu^* \wedge e_\mu} + L_{e_\nu^* \wedge e_\nu}) &= \det(\mathbb{1} - \mathrm{i}(L_{e_\mu^* \wedge e_\mu} + L_{e_\nu^* \wedge e_\nu})) \\ &= \det((\mathbb{1} - \mathrm{i}L_{e_\mu^* \wedge e_\mu})(\mathbb{1} - \mathrm{i}L_{e_\nu^* \wedge e_\nu})) \\ &= \det(\mathbb{1} - \mathrm{i}L_{e_\mu^* \wedge e_\mu}) \det(\mathbb{1} - \mathrm{i}L_{e_\nu^* \wedge e_\nu}) \\ &= \chi(L_{e_\mu^* \wedge e_\mu})\chi(L_{e_\nu^* \wedge e_\nu}).\end{aligned}$$

Therefore, $(Z|L_{e_\mu^* \wedge e_\mu})$ and $(Z|L_{e_\nu^* \wedge e_\nu})$ are independent random variables for $\mu \neq \nu$, and because of the support properties of ρ one obtains for $A = e_{\mu_n}^* \wedge \cdots e_{\mu_1}^* \wedge e_{\mu_1} \wedge \cdots \wedge e_{\mu_n}$ the following integral formula.

$$\begin{aligned}\int \langle \exp Z|A\rangle\, \mathrm{d}\rho\,(Z) &= \int \langle Z|e_{\mu_1}^* \wedge e_{\mu_1}\rangle \cdots \langle Z|e_{\mu_n}^* \wedge e_{\mu_n}\rangle\, \mathrm{d}\rho\,(Z) \\ &= \int \langle Z|e_{\mu_1}^* \wedge e_{\mu_1}\rangle\, \mathrm{d}\rho\,(Z) \cdots \int \langle Z|e_{\mu_n}^* \wedge e_{\mu_n}\rangle\, \mathrm{d}\rho\,(Z) \\ &= \langle \Omega_-|e_{\mu_1}^* \wedge e_{\mu_1}\rangle \cdots \langle \Omega_-|e_{\mu_n}^* \wedge e_{\mu_n}\rangle \\ &= \langle \exp \Omega_-|A\rangle.\end{aligned} \tag{40}$$

For general tensors $A \in \mathcal{T}_n(\mathcal{E}_1) \wedge \mathcal{T}_n(\mathcal{E}_1^*)$ the same equation follows from linearity of integral and scalar product.

Mainly due to its restricted invariance properties this measure will not be further considered.

3.3.3. Non-centred Gaussian measure The third possibility is given as follows. One considers a Gaussian measure $\nu_{\Omega_S,\mathbb{1}}$ on $(\mathcal{H}_{-1} \wedge \mathcal{H}_{-1})^{\mathbb{R}}$ with Fourier transform

$$\chi(A) = \sum_{(\mathcal{H}_{-1}\wedge\mathcal{H}_{-1})^{\mathbb{R}}} \mathrm{e}^{\mathrm{i}\langle Z|A\rangle}\,\mathrm{d}\nu_{\Omega_S,\mathbb{1}} = \mathrm{e}^{\mathrm{i}\langle\Omega_S|A\rangle - \frac{1}{2}\|A\|^2} \qquad A \in (\mathcal{H}\wedge\mathcal{H})_1^{\mathbb{R}} \tag{41}$$

that is, the non-centred Gaussian measure with mean value Ω_S and covariance $\mathbb{1}$. Then $\langle\cdot|A_1\rangle$ and $\langle\cdot|A_2\rangle$ are independent random variables if $A_1 \perp A_2$

If $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}^{\mathbb{R}}$, one gets

$$\begin{aligned}
\int \langle \exp Z|f_1 \wedge \cdots \wedge f_{2n}\rangle\,\mathrm{d}\nu_{\Omega_S,\mathbb{1}}(Z)
&= \sum_{\sigma\in S_{2n}} \frac{\epsilon(\sigma)}{2^n n!} \int \langle Z|f_{\sigma(1)} \wedge f_{\sigma(2)}\rangle \cdots \langle Z|f_{\sigma(2n-1)} \wedge f_{\sigma(2n)}\rangle\,\mathrm{d}\nu_{\Omega_S,\mathbb{1}} \\
&= \sum_{\sigma\in S_{2n}} \frac{\epsilon(\sigma)}{2^n n!} \int \langle Z|f_{\sigma(1)} \wedge f_{\sigma(2)}\rangle\,\mathrm{d}\nu_{\Omega_S,\mathbb{1}} \cdots \int \langle Z|f_{\sigma(2n-1)} \wedge f_{\sigma(2n)}\rangle\,\mathrm{d}\nu_{\Omega_S,\mathbb{1}} \\
&= \sum_{\sigma\in S_{2n}} \frac{\epsilon(\sigma)}{2^n n!} \langle\Omega_S|f_{\sigma(1)} \wedge f_{\sigma(2)}\rangle \cdots \langle\Omega_S|f_{\sigma(2n-1)} \wedge f_{\sigma(2n)}\rangle \\
&= \langle \exp\Omega_S|f_1 \wedge \cdots \wedge f_{2n}\rangle.
\end{aligned} \tag{42}$$

For general $A \in \mathcal{T}^-(\mathcal{H})$ this result again follows from linearity of an integral and scalar product.

For this measure an arbitrary tensor $\Omega_S \in (\mathcal{H}_{-1} \wedge \mathcal{H}_{-1})^{\mathbb{R}}$ can be used since the mean value of a Gaussian measure does not have to satisfy any special conditions. The requirement $\Omega_S \in (\mathcal{H}_{-1} \wedge \mathcal{H}_{-1})^{\mathbb{R}}$ is especially satisfied if Ω_S is derived from a positive, or at least self-adjoint, operator S on $\mathcal{E}$.

If $\Omega_S \in (\mathcal{H}_{-1} \wedge \mathcal{H}_{-1})^{\mathbb{R}}$ is not fulfilled, as in the case of the Dirac field, one can use the measure $\mu_{\Omega_S,\mathbb{1}}$ on the complex space $\mathcal{H}_{-1} \wedge \mathcal{H}_{-1}$, which is derived from the complex white noise as an image under the translation $Z \mapsto Z + \Omega_S$.

Complex white noise fulfils

$$\sum_{\mathcal{H}_{-1}\wedge\mathcal{H}_{-1}} \langle Z|A_1\rangle \cdots \langle Z|A_n\rangle\,\mathrm{d}\mu_{0,\mathbb{1}} = 0$$

if $A_i \perp A_j$ for $i \neq j$. Therefore, instead of (42), one now obtains

$$\begin{aligned}
\int \langle \exp Z|f_1 \wedge \cdots \wedge f_{2n}\rangle\,\mathrm{d}\mu_{\Omega_S,\mathbb{1}}(Z)
&= \sum_{\sigma\in S_{2n}} \frac{\epsilon(\sigma)}{2^n n!} \int \langle Z|f_{\sigma(1)} \wedge f_{\sigma(2)}\rangle \cdots \langle Z|f_{\sigma(2n-1)} \wedge f_{\sigma(2n)}\rangle\,\mathrm{d}\mu_{\Omega_S,\mathbb{1}} \\
&= \sum_{\sigma\in S_{2n}} \frac{\epsilon(\sigma)}{2^n n!} \int \langle Z+\Omega_S|f_{\sigma(1)} \wedge f_{\sigma(2)}\rangle \cdots \langle Z+\Omega_S|f_{\sigma(2n-1)} \wedge f_{\sigma(2n)}\rangle\,\mathrm{d}\mu_{0,\mathbb{1}} \\
&= \sum_{\sigma\in S_{2n}} \frac{\epsilon(\sigma)}{2^n n!} \langle\Omega_S|f_{\sigma(1)} \wedge f_{\sigma(2)}\rangle \cdots \langle\Omega_S|f_{\sigma(2n-1)} \wedge f_{\sigma(2n)}\rangle \\
&= \langle \exp\Omega_S|f_1 \wedge \cdots \wedge f_{2n}\rangle.
\end{aligned}$$

So $\nu_{\Omega_S,\mathbb{1}}$ and $\mu_{\Omega_S,\mathbb{1}}$ are suitable measures for the representation (34). As previously mentioned, the mean value of a Gaussian measure does not have to fulfill any special conditions. Therefore no positivity requirements for S are necessary and the doubling of degrees of freedom can be omitted.

For the massless Dirac field Kupsch [19] obtains from (32) and (33) a decompostion of the measure space into the direct sum of two independent Weyl fields, but the present ansatz does not allow for such a simplification.

4. Interaction with classical gauge fields

In the presence of a classical electromagnetic field A^μ the Lagrange density of the Dirac theory is

$$\begin{aligned}\mathcal{L} &= \overline{\Psi}(\not{D} - m)\Psi = \overline{\Psi}(\not{\partial} - \mathrm{i}\not{A} - m)\Psi \\ &= \mathcal{L}_0 - \overline{\Psi}\mathrm{i}\not{A}\Psi \\ &= \mathcal{L}_0 + \overline{\Psi}A\Psi\end{aligned}$$

with $A = -i\not{A}$. In the perturbative spirit one writes for the generating functional of the Schwinger functions

$$L(F) = \frac{\langle \exp \Omega_S | \exp \Omega_A \wedge F\rangle}{\langle \exp \Omega_S | \exp \Omega_A\rangle} \qquad F \in \mathcal{T}^-(\mathcal{H}_1). \tag{43}$$

The tensor Ω_S represents the free two-point function according to equation (27), and $\Omega_A \in \mathcal{H}_1 \wedge \mathcal{H}_1$ is defined by the interaction A through the following equation

$$\langle f|Ag\rangle = \langle \Omega_A | f \wedge g\rangle \qquad f, g \in \mathcal{H}_1.$$

According to [5], equation 3.69 (see also [8], equation (C.2)) this functional can be written as

$$L(F) = \langle \exp \Omega_T | F\rangle \tag{44}$$

with $T = (\mathbb{1}+SA)^{-1}S$. The functional (43) can be represented by the non-centred Gaussian measure $\nu_{\Omega_T,\mathbb{1}}$ or $\mu_{\Omega_T,\mathbb{1}}$, exactly as in the free field case.

The question of absolute continuity of the interacting measure $\nu_{\Omega_T,\mathbb{1}}$ relative to the free measure $\nu_{\Omega_S,\mathbb{1}}$ reduces to the same question for Gaussian measures with different mean values. For these, according to [16, ch II, theorem 5.3], one obtains absolute continuity if and only if

$$\Omega_T - \Omega_S = \Omega_{T-S} \in \mathcal{H} \wedge \mathcal{H}.$$

According to equation (12) this is equivalent to (see also [17, p 420])

$$T - S \in C_2(\mathcal{H}).$$

Since S is continuous,

$$(\mathbb{1} + SA)^{-1} - \mathbb{1} = \frac{SA}{\mathbb{1} + SA} \in C_2(\mathcal{H})$$

is a sufficient condition for absolute continuity. Assuming continuity of $(\mathbb{1} + SA)^{-1}$ one obtains another sufficient condition, namely

$$SA \in C_2(\mathcal{H}). \tag{45}$$

This result is also found in [18, section 8], but with a different method. The condition (45) is a non-trivial requirement which can be fulfilled only after a non-local UV regularization (see [18, section 8]).

Also for the complex case $\mu_{\Omega_S,\mathbb{1}}$, condition (45) remains valid since $\mu_{\Omega_S,\mathbb{1}}$ can be written as

$$\mu_{\Omega_S,\mathbb{1}} = \nu_{\Omega_S^1,\frac{1}{2}\mathbb{1}} \times \nu_{\Omega_S^2,\frac{1}{2}\mathbb{1}}$$

where $\Omega_S^{1,2}$ result from the decomposition

$$\Omega_S = \Omega_S^1 + \mathrm{i}\Omega_S^2$$

of Ω_S into real and imaginary part.

5. Conclusions

In this paper a new method for the construction of fermionic Schwinger functions as moments of a measure is presented. It essentially uses the fact that fermionic Schwinger functions are different from zero only for an even number of arguments. The free theory is treated and also the interaction with a classical electromagnetic field can be described analogueously to the free field case. But this strongly depends on the algebraic identity (44) and up till now no method has been completely elaborated to construct non-trivial measures for other models, where the interaction is of higher order in the fermionic fields, as the Gross–Neveau or Schwinger model, for example. In that case, the main problem is to find a closed formula for the functional $L(F) = \langle \exp \Omega_S | \exp \Omega_A \wedge F \rangle$, where A now is a fourth-order tensor $\Omega_A \in \mathcal{T}_4^-(\mathcal{H}_1)$. Work in this direction is currently in progress and utilizes, besides other methods, some functional analytic tools developed for the probabilistic ansatz for bosonic theories.

One last remark concerns the support properties of the measures in this paper. Instead of the Hilbert space structure $\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}$ one could also use the structure $\mathcal{S}(\mathbb{R}^4) \subset \mathcal{L}^2(\mathbb{R}^4) \subset \mathcal{S}'(\mathbb{R}^4)$ of $\mathcal{D}(\mathbb{R}^4) \subset \mathcal{L}^2(\mathbb{R}^4) \subset \mathcal{D}'(\mathbb{R}^4)$ for the case $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^4)$, which is more familiar from bosonic theory. In that case the measures are supported on the distribution spaces $\mathcal{S}'$ or $\mathcal{D}'$. The connection between this formulation and the one used in this paper is based on a special version of Minlos theorem [11, theorem 2.6.1]. It essentially states that, under certain continuity assumptions on the characteristic functional, the support of a measure on $\mathcal{S}'(\mathbb{R}^4)$ or $\mathcal{S}'(\mathbb{R}^4)$ can be restricted to a subset of $\mathcal{S}'(\mathbb{R}^4)$ or $\mathcal{D}'(\mathbb{R}^4)$, which can be identified with $\mathcal{H}_{-1}$ (see also [10, theorem 1.1]).

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